program $\begin{cases} \text{NLP} & 存在\text{ non-linear 的式子.} \\ \text{LP} & \sum_1^n a_i x_i + \beta \\ \text{IP} & \text{LP + int. constraints} \end{cases}$

$\sin(\pi x) = 0$ ← $x \in \mathbb{Z}$

$x(1-x) = 0$ ← $x \in \{0, 1\}$

列 LP. 1. 绝对值 $|a-b|$

variable : λ
obj func : min λ
constraint : $a-b \le \lambda$ $\quad b-a \le \lambda$.

2. $x = a \cdot b \quad x, a, b \in \{0, 1\}$

$$\boxed{a+b-1 \le x \le \frac{a+b}{2}}$$

LP $\sum_1^n a_i x_i + \beta$

Ferka's lamma : exactly 1 true : ① $Ax=b,\ x \ge 0$ has a sol
② $A^T y \ge 0,\ b^T y < 0$

sol $\boxed{\begin{array}{ll} \max & c^T x \\ \text{s.t} & Ax = b \\ & x \ge 0 \end{array}}$

infeasible

找 y. ($y^T A \ge 0.\ y^T b < 0$)

$A\bar{x} = b \Leftrightarrow \underbrace{y^T}_{\ge 0} \underbrace{A}_{\ge 0}\bar{x} = \underbrace{y^T b}_{<0}$ PNB

$A_B^T c_B$

feasible

optimal

找 y ($y^T b = c^T x \quad y^T A \ge c^T$)

$c^T \bar{x} - c^T \tilde{x} = \underbrace{(y^T A - c^T)}_{\ge 0} \underbrace{\tilde{x}}_{\ge 0} \ge 0$ 即可.

unbounded

找 $\bar{x}$ & d. ($Ad = 0 \quad c^T d > 0$)

Let $\tilde{x}(t) = \bar{x} + td \quad (t \ge 0)$

→ 证 $A\tilde{x}(t) = b \Rightarrow \tilde{x}(t)$ feasible

→ $\lim_{t \to \infty} \text{val}(\tilde{x}(t)) = \infty$

都.
这些含及 P&D.

SEF
canonical form

$\max \{c^T x + \bar{z} : Ax = b.\ x \ge 0\}$

LP → SEF : ① min → max ② $\ge / \le$ → = ③ - → + ④ free → ≥ 0

同时满足 $A_B = I \quad c_B = 0$

$\boxed{\begin{array}{ll} \max & (c' \quad 0)x \\ \text{s.t} & (A' \quad I)x = b' \end{array}}$

找 $y = A_B^{-T} c_B$

找 canonical form :

$\boxed{\begin{array}{ll} \max & \underbrace{(c^T - y^T A)}_{c'^T} x + \underbrace{y^T b}_{\bar{z}} \\ \text{s.t} & \underbrace{A_B^{-1} A}_{A'} x = \underbrace{A_B^{-1} b}_{b'} \\ & x \ge 0 \end{array}}$

解法 : simplex (2) :

$P \begin{cases} \text{feasible} \begin{cases} \text{optimal} \\ \text{unbounded} \end{cases} \\ \text{infeasible} \end{cases}$

Ab-1

Phase I
题目不提供 B.
$P \to P'_{\text{canonical}}$

Phase II
提供 B.
$P \to Q$

Q $\boxed{\begin{array}{ll} \max & (0 \quad -\mathbb{1}^T)x \\ \text{s.t} & (A \quad I)x = b \end{array}}$

Table 4.1 Primal–dual pairs

(P_{max})				(P_{min})	
		≤ constraint	≥ 0 variable		
max subject to	$c^\top x$	= constraint	free variable	min subject to	$b^\top y$
		≥ constraint	≤ 0 variable		
	Ax **?** b	≥ 0 variable	≥ constraint		$A^\top y$ **?** c
	x **?** 0	free variable	= constraint		y **?** 0
		≤ 0 variable	≤ constraint		

duality (3-5) :
(P & D)

weak : $c^T x \le b^T y$.
max ≤ min

$c^T \bar{x} = b^T \bar{y}$. $\underset{\text{strong}}{\overset{\text{weak}}{\rightleftarrows}}$ $\bar{x}, \bar{y}$ 为 opt sol.

$\text{val}_P(\bar{x}) \ge \text{val}_D(\bar{y})$

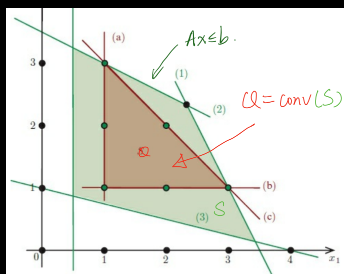
D \ P	opt	unbou	infeas
opt	√ s	X w	X s
unbou	X w	X w	√ w
infea	X s	√ w	√

CS condition : $\boxed{\begin{array}{lll} \bar{y}_i = 0 & \text{or} & \text{row}_i(A)\bar{x} = b_i \\ \bar{x}_j = 0 & \text{or} & \text{row}_j(A^T)\bar{y} = c_j \end{array}}$ $\Leftrightarrow$ $\boxed{\begin{array}{l} \bar{x} \text{ opt for } P \\ \bar{y} \text{ opt for } D \end{array}}$

IP LP + $\mathbb{Z}$ constraints

解法：(b)

$Ax\le b$ 与 IP 的 convex hull sol 的性质：



feasible $\Longleftrightarrow$ feasible

infeasible $\Longleftrightarrow$ infeasible

$\bar{x}$ is opt sol $\Longleftarrow$ $\bar{x}$ is opt sol

$\boxed{\bar{x} \text{ is opt sol} + \text{extreme pt.}}$ $\Longrightarrow$ $\bar{x}$ is opt sol

求 optimal solution：Cutting Plane. (笔记 P64)

① 把P写成canonical form

② 用 Phase I iteration 找到一个解 $\bar{x}$

③ 找 cutting plane：$\alpha_1 x_1+\alpha_2 x_2\le\beta$. (*)

s.t. 1. (*) is valid for IP

2. $\bar{x}$ not satisfy (*)

④ 在P中判断 $\bar{x}$ 是否满足条件.

- 满足：结束
- 不满足：将(*)作为原来P的下一行

canonical form 中的一个式子 → 取 "$\lfloor\ \rfloor$"

应用：Shortest Path

P is shortest path

$\Leftrightarrow$ length of P = Σ width of all st-cuts (optimality condition of s.p)

$\Leftrightarrow$ 同时满足：① all edges of P are equality edge for $\bar{y}$ (feasible width)

slackness = 0

② all active cut $\bar{y}$ contain 1 edge of P

(proposition 3.6)

$\Leftrightarrow$ (correctness of shortest path algorithm)

Algorithm 3.2 Shortest path

Input: Graph $G=(V,E)$, costs $c_e\ge 0$ for all $e\in E$, $s,t\in V$, where $s\ne t$.
Output: A shortest st-path P.

```
y_W := 0 for all st-cuts δ(W). Set U := {s}
while t ∉ U do
    Let ab be an edge in δ(U) of smallest slack for y where a ∈ U, b ∉ U
    y_U := slack_y(ab)
    U := U ∪ {b}
    change edge ab into an arc →ab
end while
return A directed st-path P.
```

(P) $x_e\ (e\in E)=\begin{cases}1 & e\in P\\ 0 & e\notin P\end{cases}$

$$\boxed{\begin{aligned}&\min \sum_{e\in E} c_e\cdot x_e\\ &\text{s.t.} \sum_{e\in\delta(u)} x_e\ge 1 \quad \forall U\subseteq V.\ s\in U\ \ t\notin U\\ &\qquad x\ge 0\end{aligned}}$$

(D) width

$$\boxed{\begin{aligned}&\max \Sigma\,(y_u : \delta(u) \text{ is st-cut})\\ &\text{s.t. } \Sigma\,(y_u : \delta(u) \text{ is st-cut}, e\in\delta(u))\le c_e. \quad e\in E\\ &\qquad y\ge 0\end{aligned}}$$

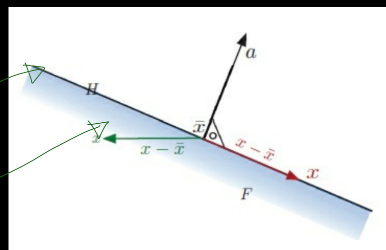
$\text{slack}_y(e)=c_e-\Sigma(y_u : \delta(u) \text{ is an st-cut contain } e)$

Geometry

$\dim(H) = n-1$

hyperplane : $\{x \in \mathbb{R}^n, a^T x = \beta\}$

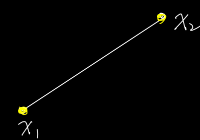
halfspace $\{x \in \mathbb{R}^n, a^T x \le \beta\}$



line segment

$\{x \in \mathbb{R}^d : x = \lambda u + (1-\lambda) v \quad 0 \le \lambda \le 1\}$

convex $\forall x_1, x_2 \in S,\ \lambda \in [0,1]$ $\lambda x_1 + (1-\lambda) x_2 = x \in S$



extreme pts $P = \{x \in \mathbb{R}^d : Ax \le b\}$ be a polyhedron. $\bar{x} \in P$. P34

↑

opt sol.

- Let $A^= x = b^=$ be set of tight constraints for $\bar{x}$.

$\bar{x}$ is extreme pt of $P \Leftrightarrow \operatorname{rank}(A^=) = \operatorname{rank}\left(\begin{bmatrix} A_N & A_B \\ -A_N & -A_B \\ -I & 0 \end{bmatrix}\right) = d$

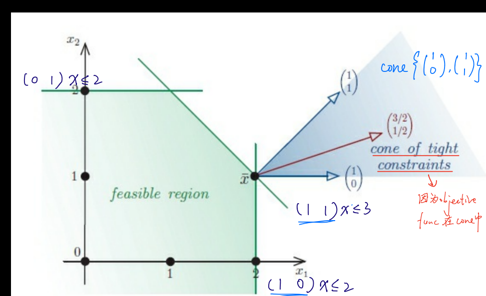
$\Leftrightarrow$ $\bar{x}$ is a basic feasible sol of $Ax = b$ (theorem 2.20)

$\bar{x}$ is not extreme pt $\Leftrightarrow$ $\bar{x} = \lambda x_1 + (1-\lambda) x_2 \ (0 < \lambda < 1,\ x_1 \ne x_2)$

cone :

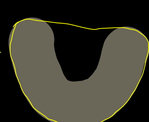
(CS)

$$\max \left(\tfrac{3}{2} \quad \tfrac{1}{2}\right) x \quad \text{s.t.} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} x \le \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$



convex hull : smallest convex set contain S

conv(S)



cutting plane

不等式* 满足：① (*) valid for IP. (所有 IP 的 sol 都满足*)

② $x^{(1)}$ 不满足*

↖ 非整数解

Algorithm 2: Cutting Plane Scheme

Input : (IP)$=\max\{c^T x : Ax \le b, x \in \mathbb{Z}\}$

repeat
- Let (P) denote $\max\{c^T x : Ax \le b\}$ (integer program relaxation)
- **if** *(P) is infeasible* **then**
  - **return** *(IP) is also infeasible*
- $\bar{x} \leftarrow$ optimal solution to (P)
- **if** *$\bar{x}$ is integral* **then**
  - **return** *$\bar{x}$ is also optimal for (IP)*
- Finding a cutting plane $a^T x \le \beta$ for $\bar{x}$
- Add a constraint $a^T x \le \beta$ to the system $Ax \le b$

until

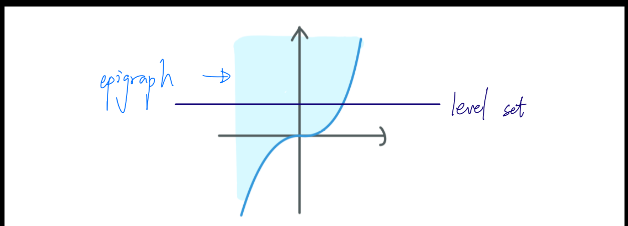
$$x_{r(i)} + \sum_{j \in N} A_{ij} x_j = b_i$$

$$x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j \le b_i$$

$$\underbrace{x_{r(i)}}_{b_i} + \sum_{j \in N} \lfloor A_{ij} \rfloor \underbrace{x_j}_{=0} = b_i \ge \lfloor b_i \rfloor$$

NLP 存在 non-linear 的式子

$$\boxed{\begin{array}{ll} \min & f(x) \\ \text{s.t} & g_1(x) \le 0 \\ & \vdots \\ & g_n(x) \le 0 \end{array}}$$



not convex function $\Leftrightarrow$ not convex epigraph

convex NLP: f.g 全是 convex function

convex function: $\forall u, v \in \mathbb{R}^n \quad \lambda \in [0,1]. \quad f(\lambda u + (1-\lambda)v) \le \lambda f(u) + (1-\lambda) f(v)$

level set: $\{x \in \mathbb{R}^n : g(x) \le \beta\}$

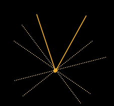
epigraph: $\text{epi}(f) = \left\{ \begin{pmatrix} \alpha \\ x \end{pmatrix} \in \mathbb{R} \times \mathbb{R}^n : f(x) \le \alpha \right\}$ epi(f) is convex set $\Leftrightarrow$ convex function

subgradient (s): $\underline{h}(x) = f(\bar{x}) + s^T(x - \bar{x}) \le f(x) \quad \forall x \in \mathbb{R}^n$

supporting

subgradient.

sol.



slater point ($\bar{x}$): $\boxed{\begin{array}{ll} \min & c^T x \\ \text{s.t} & g_i(x) \le 0 \end{array}}$ 当 $g_i(\bar{x}) < 0 \quad \forall i \in \{1, \dots, k\}$

KKT: 用于证明 feasible sol $\bar{x}$ is optimal in NLP

1. 写出 NLP 的 relaxation
2. 证明 $\bar{x}$ is opt in relaxation
3. 证明 $\bar{x}$ is opt in NLP

证明 $\bar{x} = (1,1)^T$ *is an optimal solution to*

$$\begin{array}{ll} \min & -x_1 - x_2 \\ \text{s.t.} & -x_2 + x_1^2 \le 0 \\ & -x_1 + x_2^2 \le 0 \\ & -x_1 + \frac{1}{2} \le 0 \end{array}$$

one subgradient $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\rightarrow$ relaxation

$$\boxed{\begin{array}{ll} \min & -x_1 - x_2. \\ \text{s.t} & 2x_1 - x_2 \le 1 \\ & -x_1 + 2x_2 \le 1 \end{array}}$$

$h(x) = f(\bar{x}) + \underline{\underline{s^T}}(x - \bar{x})$

$= 0 + \underline{(-1 \ \ 2)} \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$

$= -(x_1 - 1) + 2(x_2 - 1)$

$= -x_1 + 2x_2 - 1$

$\rightarrow \bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is feasible

$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is subgradient for g_1 at $\bar{x}$

$\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is subgradient for g_2 at $\bar{x}$

$-\begin{pmatrix} -1 \\ -1 \end{pmatrix} \in \text{cone}\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\} \Rightarrow \bar{x}$ is optimal